Indian Statistical Institute, Bangalore

B. Math.

First Year, Second Semester

Algebra II (Linear Algebra)

Final Examination Date : April 26, 2010

Maximum marks: 100 Time: 3 hours

1. Prove that $\mathcal{B} = \{b_1, b_2, b_3\}$ is a basis for $\mathbb{R}^3$, where

$$b_1 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, b_3 = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix}.$$

Express the vector x as a linear combination of these basis vectors, where

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

[10]

2. Let $\mathcal{G}$ be a 7 dimensional vector space over $\mathbb{R}$. Suppose $\mathcal{H}$, $\mathcal{K}$ are subspaces of $\mathcal{G}$ such that $\dim(\mathcal{H}) = 4$, $\dim(\mathcal{K}) = 5$. Show that $\dim\ (\mathcal{H} \bigcap \mathcal{K}) \geq 2$. [10]

3. Let V, W be finite dimensional vector spaces over $\mathbb{R}$. Suppose $T : V \to W$ is a linear map. Show that

$$\dim\ (V) = \dim\ (\text{Ker}\ (T)) + \dim(\text{Range}(T)).$$

(Here Ker $(T) = \{x : T(x) = 0.\}$). [20]

4. Let $M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices, considered as a vector space over the field of complex numbers. Show that

$$\langle A, B\rangle = \text{trace}(A^*B)$$

defines an inner product on $M_n(\mathbb{C})$. Obtain an ortho-normal basis for $M_n(\mathbb{C})$.
. [10]

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5. Let B be an $n \times n$ self-adjoint matrix. Show that there exists a polynomial q such that $q(B) = |B|$. (Here $|B| = (B^*B)^{\frac{1}{2}}$). [15]

6. Suppose $C \in M_n(\mathbb{C})$. Show that C decomposes uniquely as $C = A + iB$ where A, B are self-adjoint matrices. Here A is known as the real part of C and B is known as the imaginary part of C. Now show that C is normal iff A and B commute. [15]

7. Suppose $A = [a_{ij}]_{1 \leq i,j \leq n}$ is a positive semidefinite matrix and one of its diagonal entries $a_{ii} = 0$. Then show that $a_{ij} = a_{ji} = 0$ for $1 \leq j \leq n$. [10]

8. Diagonalize following matrices and obtain polar decompositions for them.

$$M = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, N = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

[20]